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ANALYSIS OF FRINGE PATTERNS BY THE METHOD OF INTEGRAL BOUNDARY EQUATIONS IN THE SOLUTION OF PLANE ELASTOPLASTIC PROBLEMS

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In experimental studies of plane problems of the mechanics of deformable bodies by moire methods [1-3] or holographic interferometry with the use of superimposed interferometers [4], the information that is obtained is represented in the form of patterns of interference fringes. By analyzing these patterns, it is possible to determine the stress and strain fields in the region being studied. There are various approaches and corresponding algorithms for solving problems [2, 5-9, etc.] based on determination of fringe-order functions N(x, y) in the region being studied, the transition from these functions to functions of the plane components of the displacements u(x, y) and v(x, y), and determination of their partial derivatives.

The fact that the strain components are calculated by differentiating reconstructed functions makes these methods highly sensitive to errors and distortions in the initial data and to the choice for the criterion of their approximation. At the same time, the information obtained from the experiment is inadequate to correctly approximate the initial functions, since it is necessary to know not only the orders of the fringes at the boundaries of the region but also their derivatives. Application packages currently available for analyzing fringe patterns [9-12] automatically sample and numerically filter the initial data, which reduces the laboriousness of the calculations considerably. However, the algorithms used for subsequent analysis still have the deficiencies noted above.

The authors of [13] noted the efficacy of synthesizing holographic interferometry and numerical potential methods to study the elastoplastic state of three-dimensional bodies. Here, to establish the stress—strain state inside the region, it is sufficient to have information that can be obtained from the fringe patterns at its boundaries. Among the advantages of this approach is the smoothing effect inherent in integral methods: the errors of the boundary conditions turn out to be considerably lower farther into the region than near the boundaries.

In the present study, we examine the feasibility of using theoretical solutions obtained by numerical realization of the method of integral boundary equations (IBE) to analyze fringe patterns in an investigation of elastoplastic fields of stress and strain.

1. Formulation of the Problem. Four fringe patterns are recorded [4] to find the plane components of the displacements u(x, y) and v(x, y) with the use of superimposed interferometers. In this case, the values of the displacements can be found from the formulas

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 164-170, March-April, 1990. Original article submitted November 11, 1988; revision submitted January 23, 1989.

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$$v = K_v(N_1 - N_2)/2; \tag{1.1}$$

$$u = K_u(N_3 - N_4)/2. \tag{1.2}$$

Here, N_1, N_2, N_3, and N_4 are the orders of the interference fringes from the respective patterns; K_v and K_u are certain coefficients dependent on the light source that is used and on the parameters characterizing the conditions under which the patterns were recorded.

When moire methods are used to study objects, the fringe patterns that are obtained are patterns of isolines of the fields of the displacements u(x, y) and v(x, y), the values of which are calculated from the formulas

$$u = N_u K_u'; \tag{1.3}$$

$$v = N_v K_v' \tag{1.4}$$

(N_u and N_v are the orders of the fringes from the corresponding moire patterns; K_u' and K_v' are scale coefficients dependent on the scanning frequency and recording conditions).

The interference patterns are analyzed in the following sequence. The region being studied is delimited by a certain smooth closed contour. The points of intersection of the interference fringes and this contour are found for each experimental pattern. Through approximation, we reconstruct the fringe-order functions $N_1(\xi)$, $N_2(\xi)$, $N_3(\xi)$, $N_4(\xi)$ or $N_u(\xi)$, $N_v(\xi)$. We then use (1.1-1.4) to determine the displacements on the boundary of the region $u(\xi)$ and $v(\xi)$ (where ξ is the coordinate of a point on the contour of the boundary of the region Ω). These displacements are then used as boundary conditions to solve the two-dimensional elastoplastic problem.

Thus, analysis of the interference patterns reduces to the numerical solution of a plane elastoplastic problem with assigned boundary conditions obtained from an experiment.

2. Numerical Solution of a Plane Elastoplastic Problem by the IBE Method. The solution is constructed within the framework of the theory of small elastoplastic strains in combination with the method of elastic solutions and use of the von-Mises–Huber yield condition [14, 15]

$$\sigma_i = \sigma_y, \tag{2.1}$$

i.e., plastic strain develops when the intensity of the stresses σ_i reaches the yield point of the material in tension σ_y. For a plane stress state

$$\sigma_i = \sqrt{\sigma_{11}^2 - \sigma_{11}\sigma_{22} + \sigma_{22}^2 + 3\sigma_{12}^2}; \tag{2.2}$$

$$\varepsilon_i = (\sqrt{2}/3)\sqrt{(\varepsilon_{11} - \varepsilon_{22})^2 + (\varepsilon_{22} - \varepsilon_{33})^2 + (\varepsilon_{33} - \varepsilon_{11})^2 + 1.5\varepsilon_{12}^2} \tag{2.3}$$

(ε_i is the strain intensity).

The method of elastic solutions is a method of successive approximation in which a normal problem of the theory of elasticity is solved at each step. It is known that the basic equations of the theory of plasticity can be written in a form which is analogous to the corresponding equations of the theory of elasticity, with allowance for the action of certain additional body ψ_i and surface T_i forces [14]. These forces are applied in the region of a plate and are determined as follows: for points inside the region

$$\psi_i = \partial\sigma_{ij}^0/\partial x_j \quad (i, j = 1, 2); \tag{2.4}$$

for points on the boundary of the region

$$T_i = \sigma_{ij}^0 n_j \qquad (i, j = 1, 2). \tag{2.5}$$

Here, $\sigma_{ij}{}^0 = \sigma_{ij}{}^* - \sigma_{ij}$ are components of the additional stresses calculated as the difference between the stresses in the elastoplastic medium $\sigma_{ij}{}^*$ and the stresses in the analogous elastic region σ_{ij}; n_j are the direction cosines of an external normal to the boundary of the region.

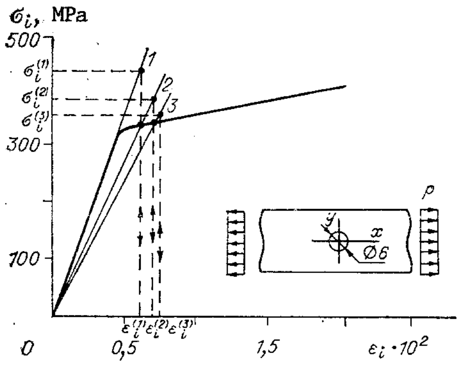

Fig. 1

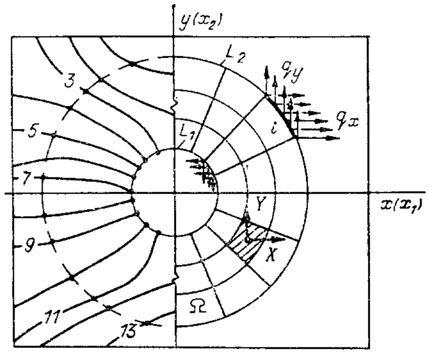

Fig. 2

We have the following [15] for the stress components in an elastoplastic medium

$$\sigma_{11}^{*}=2G^{*}[\varepsilon_{11}+\mu^{*}\theta/(1-\mu^{*})],$$
$$\sigma_{22}^{*}=2G^{*}[\varepsilon_{22}+\mu^{*}\theta/(1-\mu^{*})],\quad \sigma_{12}^{*}=G^{*}\varepsilon_{12}, \tag{2.6}$$

where $\theta = \varepsilon_{11} + \varepsilon_{22}$; $G^* = \sigma_i/3\varepsilon_i$ is the variable shear modulus; $\mu^* = (0.5 - 3\lambda G^*)/(1 + 3\lambda G^*)$; $\lambda = (1 - 2\mu)/E$; E and μ are the elastic modulus and Poisson's ratio of the material of the elastic medium. The functional dependence of σ_i on ε_i should also be known to solve the problem. Figure 1 presents the stress–strain curve of the material $\sigma_i = f(\varepsilon_i)$.

As the initial approximation, we take the additional loads to be equal to zero and we solve the plane problem of the theory of elasticity. We determine the parameters of the stress–strain state σ_{ij} and ε_{ij} and the values of σ_i and ε_i. For the plastic region ($\sigma_i > \sigma_y$), we use the stress–strain curve $\sigma_i = f(\varepsilon_i)$ to find the values of σ_i^* that correspond to ε_i. We then use Eqs. (2.6) to calculate the stress components σ_{ij}^*. This is followed by the use of (2.4) and (2.5) to calculate the additional body and surface forces. In the second approximation, we solve the initial boundary-value problem of the theory of elasticity with allowance for the action of the additional forces. The computing process is then repeated again. It is ended when the difference between the results of two successive approximations is either sufficiently small or turns out to be within an acceptable range of accuracy. Calculations we performed showed the rapid convergence of the process – two or three iterations were sufficient in many cases.

Thus, the solution of the plane elastoplastic problem reduces to the solution of a certain sequence of linearly elastic problems with allowance for additional body and surface forces applied in the plastic region.

Two-dimensional linearly elastic problems are solved here by numerical realization of the method of integral boundary equations in accordance with the approach proposed in [16]. An assigned region Ω is examined as part of an infinite plate with unknown distributed loads q_x and q_y acting along the contours of its boundaries. It is also necessary to find values of q_x and q_y at which the specified conditions will be satisfied on the contours L_1 and L_2.

The boundary of the region is divided into n sufficiently small sections within which we can assume that q_x = const and q_y = const. Using the solution for a concentrated force acting in a plane, by integrating over each section and summing over the contours of the boundaries we can obtain expressions for the stresses and the displacements at points of the region which include the boundary. If concentrated forces act inside the region, then the necessary terms are added to these expressions.

The stresses and the displacements from the action of the concentrated force P in the direction of the x axis are found as

$$\sigma_x=\frac{P}{4\pi}\frac{x}{x^2+y^2}\left[-(3+\mu)+2(1+\mu)\frac{y^2}{x^2+y^2}\right],$$
$$\sigma_y=\frac{P}{4\pi}\frac{x}{x^2+y^2}\left[1-\mu-2(1+\mu)\frac{y^2}{x^2+y^2}\right], \tag{2.7}$$

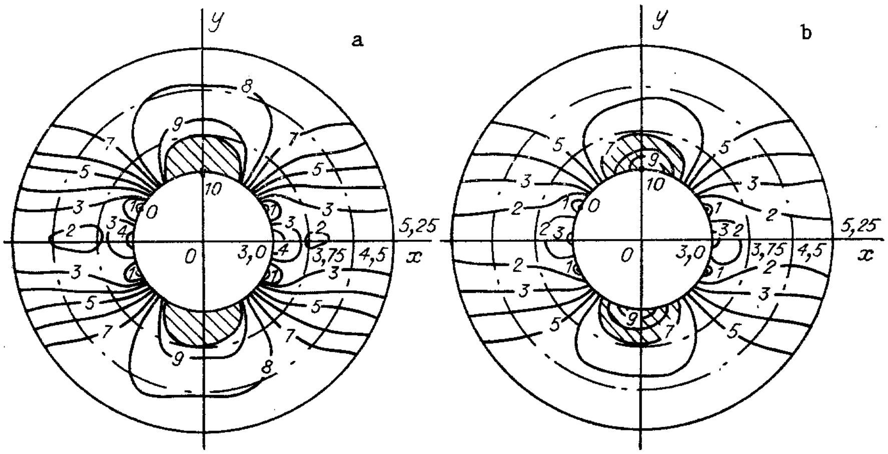

Fig. 3

$$\tau_{xy} = -\frac{P}{4\pi}\frac{xy}{x^2+y^2}\left[1-\mu+2(1+\mu)\frac{x^2}{x^2+y^2}\right], \tag{2.7}$$
$$u = -\frac{P}{8\pi G}\left[\frac{3-\mu}{2}\ln(x^2+y^2)+(1+\mu)\frac{y^2}{x^2+y^2}\right], \quad v = \frac{P}{8\pi G}(1+\mu)\frac{xy}{x^2+y^2}$$

(G is the shear modulus; x and y are the coordinates of the point where the sought quantities are determined relative to the point of application of the force; u and v are the displacements of points in the direction of the x and y axes).

The displacements and stresses from the distributed loads q_x and q_y in a given section for points located outside the section are calculated from Eqs. (2.7) on the basis of the concentrated force, the latter being the resultant of these loads within the section of their application. The components of the stresses and the displacements for a point belonging to the section of application of q_x and q_y are found from the total action of the system of concentrated forces, which equivalently replaces a uniformly distributed load. In determining the stresses in this case, we add to the calculated values additional terms which account for singularity. These additional terms have the following form for the distributed load q_x

$$\begin{aligned} \sigma_x &= \pm(q_x/2)\cos\alpha[1+(1+\mu)\sin^2\alpha], \\ \sigma_y &= \mp(q_x/2)\cos\alpha(\sin^2\alpha-\mu\cos^2\alpha), \\ \tau_{xy} &= \pm(q_x/2)\sin\alpha(\sin^2\alpha-\mu\cos^2\alpha) \end{aligned} \tag{2.8}$$

(α is the angle formed by a normal to the contour of the boundary at the point being considered and the x axis).

In order to discretize the additional body forces, we use a coordinate grid to subdivide the region being studied Ω into subregions or elements. We check for the satisfaction of condition (2.1) for each element and we localize the plastic strain zone. At the center of gravity of an element belonging to this zone, we apply concentrated forces X and Y (Fig. 2) equivalent to the action of the additional body forces. The values of X and Y can be found directly from the equilibrium condition for this element when the additional stresses $\sigma_{ij}{}^0$ act on its boundary.

Expressions for the displacements containing the additional body forces X and Y and the unknowns q_x and q_y are introduced into the boundary conditions of the problem written for the midpoints of the sections of the contours of the boundary. We have a system of 2n equations which are linear relative to q_x and q_y. The solution of this system gives us the compensating loads q_x and q_y and then yields the stresses and displacements on the boundary and inside the specified region Ω.

3. Example. We examined the problem of determining the stress and strain fields in a uniaxially tensioned plate with a central circular hole of radius R = 3 mm. The material of

TABLE 1

Loading stage	Load, MPa	Strains															
		$\varepsilon_r \cdot 10^{-4}$				$\varepsilon_\theta \cdot 10^{-4}$				$\gamma_{r\theta} \cdot 10^{-4}$				$\varepsilon_i \cdot 10^{-4}$			
		R/R_0															
		1,0	1,25	1,5	1,75	1,0	1,25	1,5	1,75	1,0	1,25	1,5	1,75	1,0	1,25	1,5	1,75
1	65,7	—9,8 (—11,0)	—3,6 (—3,3)	—2,0 (—1,5)	—1,38 (—1,2)	30,8 (33,3)	21,2 (20,5)	17,6 (15,6)	15,6 (13,5)	0,02 (0)	0,5 (0)	0,5 (0)	0,58 (0)	27,3 (29,5)	18,5 (17,6)	15,3 (13,5)	13,6 (11.8)
2	131,4	—19,2	—7,2	—4,2	—3,1	60,1	41,3	34,4	30,4	0,02	0,6	0,6	0,6	54,2	36,1	29,9	26,5
3	184,0	—26,5	—12,2	—10,1	—8,7	78,5	53,4	47,1	41,9	0,4	0.6	0,6	0,6	70,4	46,7	41,1	36,6
4	228,6	—32,3	—10,6	—14,0	—15,8	100,0	62,0	60,2	57,4	—1,1	1,6	1,7	2,6	88,5	54,0	52,7	50,4

TABLE 2

Loading stage	Load, MPa	Stresses, MPa											
		σ_r				σ_θ				σ_i			
		R/R_0											
		1,0	1,25	1,5	1,75	1,0	1,25	1,5	1,75	1,0	1,25	1,5	1,75
1	65,7	2,3 (—0,5)	22,5 (22,9)	25,5 (24,5)	25,3 (21,8)	184,1 (198,3)	133,8 (128,0)	113,4 (100,5)	101,5 (87,5)	183,0 (198,3)	124,2 (118,2)	103,1 (90,7)	91,5 (79,0)
2	131,4	2,5	42,6	47,7	46,4	336,6	260,0	220,3	196,6	315,0	242,1	200,9	178,2
3	184,0	15,3	36,3	36,2	34,4	345,1	329,8	292,1	260,9	337,7	313,3	275,8	245,5
4	228,6	31,3	66,2	42,3	24,3	376,1	347,6	335,8	324,4	361,5	319,7	316,9	313,1

the plate was alloy D16T, with the parameters $E = 6\cdot10^4$ MPa and $\mu = 0.31$. The geometric dimensions of the specimen: width 40.2 mm, thickness 1.06 mm. The stress–strain curve of the material is shown in Fig. 1. The yield point corresponds to the values $\sigma_y = 310$ MPa and $\varepsilon_y = 0.005$.

The initial data for the numerical calculation was in the form of fringe patterns obtained in [4] with the use of superimposed interferometers. The specimen was loaded in stages. The stages corresponded to external loads p = 2.8, 5.6, 7.8, and 9.74 kN. Here, the normal stresses in the sections without a concentrator were 65.7, 131.4, 184.0, and 228.6 MPa. The holograms were recorded by the double-exposure method – at the beginning and end of each stage. The boundaries of the ring-shaped region that was studied were the contour of the hole L_1 and a circle of radius 2R which was coaxial to the hole – the contour L_2 (see Fig. 2). The functions of fringe order at the boundaries of the region were approximated in accordance with [8], and the increments of the displacements were found from Eqs. (1.1) and (1.2) for each loading stage. The total displacements were calculated by adding these increments over all previous loading stages.

The region and its boundaries were discretized with the use of a coordinate grid formed by a family of concentric circles and radial lines drawn with the same angular spacing.

A program to determine the fields of the displacements (u, v), stresses (σ_x, σ_y, τ_{xy}, and σ_r, σ_θ, $\tau_{r\theta}$), and strains (ε_x, ε_y, γ_{xy} and ε_r, ε_θ, $\gamma_{r\theta}$) and the stress (σ_i) and strain (ε_i) intensities was written in the language PL-1 for an ES computer. The results of the calculations were printed in the form of tables of values and diagrams of isolines of the respective parameters, with an indication of the plastic strain zone.

Figure 3 shows diagrams of isolines of the fields of σ_i (*a*) and ε_i (b) obtained for the third loading stage. Within the limits of the region being studied, the functions σ_i and ε_i are determined by lines corresponding to identical levels and numbered from 0 to 10. A change in the number by unity denotes increments in the functions $\Delta\sigma_i = 33.6$ MPa and $\Delta\varepsilon_i = 0.7\cdot10^{-3}$. The maximum value corresponds to line 10 (max $\sigma_i = 338.7$ MPa, max $\varepsilon_i = 7.05\cdot10^{-3}$), while the minimum value corresponds to line 0 (min $\sigma_i = 3.01$ MPa, min $\varepsilon_i = 0.5\cdot10^{-4}$). The hatched region corresponds to the plastic zone of the material. In the given case, each boundary was discretized by 36 boundary elements, while the region that was examined was subdivided into 216 internal elements. The absence of external loads was taken as the boundary conditions for the internal contour in the solution of the problem.

Tables 1 and 2 show values of the stresses and strains in the section x = 0 for four loading stages. Also shown for the first stage are the corresponding values of the parameters from the known solution of the Kirsch problem for an infinite elastic region with a circular hole. Good qualitative agreement was obtained between the isolines for the stress and strain fields and the values of the investigated parameters. The accuracy of these values was sufficient for engineering purposes.

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DETERMINATION OF STRESS INTENSITY FACTORS AT THE TIPS OF CRACKS GROWING FROM LOADED HOLES IN FINITE ANISOTROPIC PLATES

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UDC 539.43:621.8

In pin, bolt, and rivet joints, stress concentration in combination with fretting between the fastening element and the surface of the hole may lead to the formation of damages and defects. In order to be able to predict the safe life of a structure, it is necessary to be able to precisely calculate the limit load and estimate the growth of defects near fastener holes in such joints. A survey of the studies done in this area for isotropic elastic plates can be found in [1, 2], for example. Progress is being made relatively slowly in regard to the investigation of the problem for plates made of composite materials (see the surveys in [3-5], for example). The reason for this is a shortage of information on the effect of the anisotropy of the material, the boundaries of the plate, and the type of load transmission on the stress intensity factors (SIF) at the tips of cracks near loaded holes.

In the present study, we construct special representations of the solution of problems involving determination of the elastic equilibrium of a finite rectilinear anisotropic plate with a system of through slits and a loaded elliptical hole. Automatic satisfaction of the boundary conditions at the contour of the hole makes it possible to reduce the problem to the solution of a system of integral equations (IE) whose order is one less than the number of components of the boundary of the region. The absence of an unknown function at the boundary of the hole makes it possible to more efficiently find numerical solutions. Using the example of a rectangular plate with cracks originating from the contour of a hole loaded through a pin, we study the effect of anisotropy of the material, a wide range of pin-joint geometries, and different combinations of load transmission from the pin and seat with interference on the value of the SIF at the tips of the cracks. Data for an isotropic material is obtained by taking the limit in the anisotropy parameters in a numerical solution.

We will examine an elastic, rectilinearly anisotropic plate of constant thickness h bounded by closed contours Λ (an ellipse with the semiaxes a and b) and L_0 (smooth longitudinal external contour) and having n smooth internal through slits (cracks) L_j ($j = \overline{1, n}$). The plate is loaded by a self-balanced system of external forces applied to L_0 and Λ. The edges of the slits $L' = \bigcup_{j=1}^{n} L_j$ are not loaded. We will make the axes of symmetry of the ellipse coincide with the axes of the Cartesian coordinate system xOy. As the positive direction on L_0 we take the direction which leaves the plate on the left. On the slit L_j, with ends a_j and b_j, the positive direction leads from a_j to b_j. We direct the normal n to the

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 171-177, March-April, 1990. Original article submitted January 23, 1989.

